Controlled capture of a continuous vorticity distribution

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Abstract

We develop a chaotic control and targeting scheme to capture and stabilize a concentrated vortex around a cylindrical body embedded in an open fluid flow. We demonstrate that the point vortex-based control model is also valid in a continuum fluid framework, by simulating the system using a Navier-Stokes-like dynamics.

The interaction of fluid vorticity with embedded structures is one of the central problems of fluid mechanics. In particular, the interaction between bluff bodies and concentrated vorticity (e.g. shedding from a cylinder) is of major practical importance, relevant to hydro- and aerodynamics, as well as structural engineering problems [1–4]. In recent years, with the development of the theory of nonlinear dynamics, these systems have also become of fundamental interest, since chaotic behaviour may provide an intermediate regime between laminar and turbulent flow. Many numerical and experimental examples exist of the generation of chaos in such simple fluid/body systems, and they provide considerable insight into the behaviour of more complex systems [5–9].

Because simulation of a complex flow in a realistic representation (i.e. Navier–Stokes, or NS) can often be numerically intractable, a major pre-occupa-

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Letter, we demonstrate such flow modification by constructing an active nonlinear control scheme, based on the point vortex model, which can stably capture a concentrated vortex near the cylinder in the continuum representation of the system. This scheme may be a first step towards applications in important physical systems such as the dynamics of cables, piers, and structures, and also to the development of high-lift aerodynamic bodies [12,13].

Our control algorithm is a modification of the chaotic control scheme developed by Ott, Grebogi, and Yorke [14]. Briefly, this scheme (OGY) attempts to modify the behaviour of a dynamical system by exploiting the natural dynamics inherent in a chaotic regime, by actively stabilizing the system evolution around an unstable low-order periodic trajectory. The inherent advantage of this technique is that desirable dynamical states can be achieved with only small perturbations of the system. This method has been widely applied to date, and has even been used to successfully modify the flow in continuum fluid simulations (see, e.g., Ref. [15]). Here, our approach differs slightly since we deal with an open system, which technically does not possess a chaotic attractor, and arbitrary initial conditions need not ever approach the desired chaotic saddle. In this case, one can either investigate a large number of initial conditions, as has previous work on controlling transient chaos [16], or one can develop a targeting algorithm to drive a large fraction of initial conditions to the fixed point vicinity. In the following discussion, we choose the latter.

The OGY control scheme is naturally applicable to the vortex-cylinder system [17], since the Hamiltonian model possesses unstable fixed points near the cylinder, which are also exhibited in NS-like simulations [11]. Here, our general strategy will be as follows: we implement an OGY control algorithm which can stabilize (i.e. capture) a point vortex around one of the unstable fixed points of the Hamiltonian model, using only small flow perturbations. In this case, the perturbation will be an additional time-varying circulation added to the cylinder, which mimics cylinder rotation. We then use this scheme as our control model in the NS-like simulations, and attempt to demonstrate controlled capture in the continuum framework. For the purposes of this paper, we will discuss only free slip boundary conditions on the cylinder (no secondary shedding), for immediate comparison to the Hamiltonian case. No-slip simulations are in progress.

The physical system in the Hamiltonian model consists of a solid circular boundary centered at the origin, embedded in a constant uniform flow field of an inviscid fluid, with background flow velocity \( u_0 \) parallel to the x-axis, and a point vortex of circulation \( \kappa \), which is advected past the cylinder from plus to minus infinity. Additional complexity is included by assuming the cylinder has a (possibly varying) circulation \( \kappa_0 \) about its boundary, as a first approximation to modeling physical rotation of the cylinder. In non-dimensional polar coordinates \( (\rho, \phi) \) scaled by the cylinder radius \( a \) and the background flow velocity \( u_0 \), the Hamiltonian for the vortex system [10] is given by

\[
H = -\left( \frac{\sin \phi}{\rho} \right) (\rho^2 - 1) + \frac{1}{2} \sigma \ln(\rho^2 - 1) - \sigma_0 \ln \rho, \quad (1)
\]

where \( \sigma = \kappa / 2 \pi a u_0 \) and \( \sigma_0 = \kappa_0 / 2 \pi a u_0 \) are the only non-dimensional parameters required to characterize the system.

The equations of motion are calculated using the Hamilton relations and are given by

\[
d\rho/dt = -\cos \phi \left( 1 - 1/\rho^2 \right)
\]

and

\[
d\phi/dt = \left( \frac{\sin \phi}{\rho} \right) \left( 1 + 1/\rho^2 \right) - \sigma \left( \frac{1}{\rho^2 - 1} \right) + \sigma_0 / \rho^2. \quad (2)
\]

A bifurcation analysis of these equations in \( \sigma \) and \( \sigma_0 \) reveals the existence of a multitude of possible flow topologies which are determined by the number and type of fixed points in the phase plane [10]. Small perturbations of this system are known to generically produce chaos in the vortex behaviour, resulting in rich dynamical properties and a chaotic capturing phenomenon, which have also been demonstrated in a NS-like framework identical to the one employed here.

To develop our control strategy, we will choose the simplest system (no external perturbations) with fixed \( \sigma = -2.96296 \), and use as our control parameter \( \sigma_0 \), hence \( \sigma_0 = \sigma_0(t) \), and \( \sigma_0 = 0 \) when the vortex is at infinity. This produces a flow topology with a single zero of the physical flow field at \( \rho = 3.0 \), \( \phi = -\frac{1}{2} \pi \), which has a corresponding un-
stable (hyperbolic) fixed point in the phase space at these coordinates [10]. Note that interpretation of the flow topology for point vortex systems is particularly straightforward, since the phase space is simply a re-scaled version of the configuration space. We seek to capture and stabilize the passing vortex at this fixed point, using only small $\sigma_o(t)$, by manipulating the positions of the stable and unstable manifolds of the hyperbolic point. Following OGY, we linearize the dynamical system about the fixed point, and calculate the movement of the invariant manifolds produced by changing $\sigma_o(t)$. Here, however, we utilize a continuous time perturbation scheme which allows corrections at arbitrary times, rather than the original OGY scheme which performs corrections only at discrete intersections of some Poincaré section. Note that this control scheme is valid for either chaotic or regular vortex behaviour, i.e. it does not explicitly rely on chaos-generating perturbations to this system.

Let $dr/dt = A(r, \sigma_o)$ represent the dynamical system of Eq. (2), and $r_0$, one of the hyperbolic fixed points. Then in a neighborhood of the fixed point $r = r_0 + \delta r_0$, the vortex motion is described by

$$d\delta r(t)/dt = J(r_0, \sigma_o)\delta r(t) + G(r_0)\delta \sigma_o(t),$$

(3)

where $J = \partial A/\partial r$, i.e. the Jacobian, and $G = (\partial A/\partial \sigma_o)$. After a short evolution time $\Delta t \ll 1$, the new incremental vortex position becomes $\delta r(t + \Delta t) = (d\delta r(t)/dt)\Delta t + \delta r(t)$, or

$$\delta r(t + \Delta t) \approx [1 + J(r_0, \sigma_o)\Delta t]\delta r(t) + G(r_0)\delta \sigma_o(t)\Delta t$$

(4)

To achieve control of the vortex trajectory using small $\sigma_o(t)$, we impose the condition that the vortex after time $\Delta t$ lies on the stable manifold of $r_0$, or rather that the local projection along the direction of the contravariant unstable eigendirection $f_u$ is zero. Hence, $\delta r(t + \Delta t) \cdot f_u = 0$. From this condition, one easily finds that the variation of the control parameter must be

$$\delta \sigma_o(t) = -(\lambda_u + 1/\Delta t)f_u \cdot \delta r(t)/f_u \cdot G$$

(5)

where $\lambda_u$ is the eigenvalue corresponding to $f_u$. In practice, $\lambda_u$ and $f_u$ can be directly computed analytically from the linearization of Eq. (2) by eigenvalue analysis. To remain within the linearized regime, we typically set an upper bound $\sigma_o^*$ within which $\delta \sigma_o$ is allowed to vary, i.e. $|\delta \sigma_o| < \sigma_o^*$.

The scheme described above is sufficient to stabilize the point vortex at the hyperbolic point for all time, given that the vortex initially starts near the fixed point (i.e. the linear approximation is still valid). In an open flow such as this system, however, the probability that an advecting vortex passes sufficiently close to the fixed point to achieve control can be quite small. Hence, we further modify our scheme by adding a targeting algorithm. A simple targeting algorithm can be defined by adjusting the perturbation scheme of Eq. (5) so that the vortex is driven towards the stable manifold even when the vortex is possibly far upstream of the body. Such a scheme must have a bounded magnitude for $\sigma_o(t)$ even as $r(t) \to \infty$, and must reduce to Eq. (5) when the vortex is near the fixed point. With these restrictions in mind, we write

$$\delta r(t + \Delta t) \cdot f_u = (1 - \beta)\delta r(t) \cdot f_u, $$

(6)

with $0 < \beta < 1$. Intuitively, this scheme reduces the projection of the vortex position on $f_u$ at each control correction by the factor $1 - \beta$. We then choose the parameter $\beta$ to be a function of the distance from the fixed point, $\beta = \beta(|\delta r|)$, so that as $|\delta r| \to 0$ we must have Eq. (6) $\to \delta r(t + \Delta t) \cdot f_u = 0$. Here, we choose $\beta(|\delta r|) \sim \exp(-\alpha |\delta r|)$, although the qualitative results of the targeting scheme are not strongly dependent on this functional form. In the numerical experiments presented in this Letter, we typically choose $\alpha = 1$, however in an experimental situation $\alpha$ and $\beta$ could be chosen to reflect reasonable physical limitations of the apparatus. One should note that for this simple targeting scheme, the vortex is made to approach the fixed point along the direction of its local linearized stable eigenvector, rather than the actual stable manifold, which generally do not coincide. In this sense, a more efficient targeting scheme could be developed by first driving any initial condition to the stable manifold; such a scheme is currently under development.

Since the vortex systems under study are open, a fraction of the initial conditions will always escape to infinity for any finite control perturbation $\sigma_o$. The fraction of initial conditions which are successfully stabilized, using our control and targeting scheme,
Fig. 1. Physical system for the point vortex model, showing cylinder and sample vortex flow lines (arrows indicate direction). Small circles indicate time evolution of a point vortex under the dynamics of the control and targeting scheme. Initial vortex position (Euclidean coordinates) is (3.0, -4.0), and $\sigma = -2.96296$.

are a function of the maximum allowed perturbation size $\sigma^*_0$ (and hence $\alpha$ and $\beta$), and surround the stable manifold upstream of the cylinder. The width of this region grows significantly, even for small $\sigma^*_0$, as one moves upstream of the cylinder, hence large values of $\sigma^*_0$ are typically not necessary. Additionally, we note that if $\sigma^*_0$ remains too large, the topology of the phase space is no longer locally similar to that assumed for the desired fixed point. Hence, in practice, it is neither desirable nor necessary that $\sigma^*_0$ be chosen too large.

To demonstrate these ideas, and as a check of the control scheme, we perform a numerical experiment using only the point vortex system. Fig. 1 shows, the point vortex is rapidly brought to the fixed point, where it remains indefinitely.

That the point vortex system exhibits well-behaved control is not surprising, since we assume perfect information about the system, and the dynamical model is exact. Now, however, we implement the above scheme in a continuum fluid framework, to investigate its validity in a more realistic system. To do so, we use a pseudo-spectral evolution scheme for the NS equation, which is identical to the scheme discussed extensively in Ref. [11]. Briefly, this scheme uses a vorticity-stream function representation ($\omega$, $\psi$) of the flow, implemented on a polar coordinate grid centered on the body, which uses a finite-difference approximation in the radial direction, and a spectral decomposition in the angular direction. Following Ref. [11], the velocity field in cylindrical coordinates ($\rho$, $\phi$) is given by

$$V_\rho = (1/\rho)\partial\psi/\partial\phi, \quad V_\phi = -\partial\psi/\partial\rho,$$

for the radial and tangential velocities respectively. The dimensionless equations are written as

$$\partial\omega/\partial t = (1/\rho)\left(\frac{\partial\psi}{\partial \rho}\frac{\partial \omega}{\partial \phi} - \frac{\partial \psi}{\partial \phi}\frac{\partial \omega}{\partial \rho}\right) + \left(1/\text{Re}\right)\nabla^2\omega, \quad (7)$$

$$\omega = -\nabla^2\psi, \quad (8)$$

and the Reynolds number $\text{Re} = au_0/\nu$, with $\nu$ being the kinematic viscosity. In practice, the dynamically interesting behaviour for this system lies in the vicinity of the cylinder, but the computational domain must be extended to large $\rho$, to reduce edge effects for the long-ranged logarithmic potential. Hence, we introduce a radially stretched coordinate system, by defining a new radial coordinate $\eta$ such that $\eta = \log(\rho - 1 + b)/b$, where $b$ is the stretching parameter. Also, since the flow field is largely due to a background potential flow and the added cylinder circulation, we rewrite the NS equations to evolve only the correction to this flow. The resulting NS equations are somewhat lengthy and are omitted here for brevity, however the equations are identical to those in Ref. [11] and the reader is referred there. Finally, time evolution is achieved using a third-order Runge–Kutta scheme.

In addition to the dynamical equations, it is necessary to impose boundary conditions for ($\omega$, $\psi$) on
At the domain edge, \( \eta \to \infty \), we impose the standard boundary conditions of zero radial derivatives. On the cylinder boundary, we choose "free-slip" boundary conditions (i.e., the boundary is a streamline), hence \( \psi = 0 \) and \( \partial \omega / \partial \rho = 0 \) when \( \eta = 0 \). This boundary condition on the cylinder is typical for inviscid simulations, and we utilize it here in order to make a direct comparison with the point vortex case. However, we still retain viscosity in Eq. (7) for numerical stability, and to observe its influence on the overall dynamics. Nevertheless, the diffusive time scale is much longer than the typical eddy turn-over time (on the order of \( Re^{1/2} \)), so this NS-like dynamics is closer to an inviscid flow simulation. Physically, these conditions neglect the production of secondary vorticity, which can affect the dynamics of the concentrated vortex and possibly de-stabilize the control scheme. We are currently conducting simulations using no-slip conditions, however, and preliminary results indicate that the control scheme is still valid.

To test the validity of the control scheme in the continuum framework, we performed numerical experiments using the above NS-like evolutionary scheme, with the point vortex system as the control model. We choose flow parameters \( u_0, \sigma, \) and \( \sigma_0 \) to exactly correspond to the point vortex experiment of Fig. 1. For the NS scheme, we typically choose a 128 by 128 grid, with radial stretching parameter \( b = 0.1 \), an outer radius extending to \( \rho = 100 \), and \( Re = 1000 \). We then conduct the following experiment: we construct a blob of relatively concentrated vorticity with a Gaussian profile, with initial width measured by the standard deviation \( \delta \) of the profile. An initial vorticity is chosen by setting the total circulation equal to the point vortex case. We place the blob at an initial condition \( (x, y) = (3.0, -3.0) \), and \( \delta = 0.25 \). We then first evolve the vorticity profile with no control scheme, as shown in Fig. 2, where the contours indicate the vorticity at \( \frac{1}{3} \omega_{\text{max}} \). This corresponds to the upper trajectory of Fig. 2, and as can be seen, the blob advects past the upper side of the cylinder, following a flow line to infinity. We then repeat the experiment with exactly the same parameters, except that the control scheme is now turned on. For the purposes of the control, we replace the point vortex position with the calculated center of the continuous vorticity distribution. The second trajectory of Fig. 2 shows this result, and now the vorticity profile smoothly moves to the corresponding fixed point of the Hamiltonian system, where it remains for at least 30 characteristic times (where the simulation was terminated). Note that this result demonstrates several points: firstly, that the underlying topology of the Hamiltonian flow is still exhibited in the continuum case; secondly, that the control scheme based on the point vortex model is still valid, due to the correspondences in the topologies; and thirdly, that because the perturbations \( \sigma_0(t) \) can typically be small, the vorticity profile remains coherent, and the point vortex approximation remains valid (the correspondence is related to the symmetry of the profile [19]).

We now perform a second experiment for the continuum system. Here, we investigate the effects of finite core size in the continuum system, by choosing a wide initial vorticity profile with \( \delta = 0.6 \). We use here corresponding parameters and initial conditions to the experiment of Fig. 1, and evolve...
Fig. 3. Evolution of controlled vorticity distribution for initial conditions \((3.0, -4.0)\), \(\sigma = -2.96296\), and size \(\delta = 0.6\). One contour level corresponding to \(\frac{1}{4}\omega_{\text{max}}\) is plotted every unit of time. Dashed lines correspond to additional vorticity contours of 0.3 and 0.1 of \(\omega_{\text{max}}\) at \(t = 30\). The plot indicates that the control scheme is stable even for large, non-symmetric vorticity distributions, and still corresponds well with the Hamiltonian system.

This large vorticity profile as shown in Fig. 3. As can be seen, the large profile again moves smoothly to the Hamiltonian fixed point, where it again remains for at least 30 characteristic times. Note that even though the profile is of comparable size to the cylinder radius \(a\) (broken lines indicate contours for 0.3 and 0.1 of \(\omega_{\text{max}}\)), and is non-symmetric, the point vortex model remains valid, and the control scheme remains effective.

To indicate the dynamics of the control scheme, we plot in Fig. 4, the time history of the perturbation magnitudes \(\sigma_\delta(t)\) for three cases with the same initial conditions corresponding to Fig. 3, but varying blob sizes. All three cases show large initial magnitudes as the targeting scheme is invoked and the vortices are driven towards the stable manifold far from the fixed point. As the vortices near the fixed point, the control scheme reduces to the normal linearized scheme, and the perturbation magnitudes rapidly approach zero for the point vortex case (solid line), and for a profile with \(\delta = 0.25\) (dashed line). For a larger profile with \(\delta = 0.6\) (dash-dotted line), the perturbation magnitude asymptotically remains non-zero, resulting from a multi-pole correction to Hamiltonian dynamics.

Fig. 4. Time evolution of \(\sigma_\delta(t)\) for controlled vortex trajectories corresponding to initial conditions of Fig. 3, for different vorticity core sizes. Three different lines indicate the point vortex trajectory, the trajectory of distribution with \(\delta = 0.25\), and the trajectory of distribution with \(\delta = 0.6\). The line for \(\delta = 0.6\) is asymptotically non-zero due to the discrepancy between the dynamics of the large non-symmetric core and the point vortex model.
the point vortex model due to the non-symmetric continuous distribution. Note that regardless of this discrepancy, the control scheme remains effective, the large vortex is readily stabilized, and the perturbation magnitudes remain small.

Robustness to noise is an important property of control schemes, and we have investigated this for our scheme in two contexts. We have simulated the effects of non-uniform background fluid flow by adding a random Gaussian-distributed perturbation of varying magnitudes to the flow, i.e., \( u_0 \rightarrow u_0 + \Delta u_0 \), to simulate weak turbulence. We have also simulated measurement noise, by adding a randomly distributed perturbation to the estimate for the center-of-vorticity in the control loop. The qualitative effects of both types of noise are the same, with the result being that the control scheme is robust to significant perturbation (at least 10% in each variable). No simple estimate of maximum noise magnitude for stability is possible, since larger noise perturbations effectively only result in larger control corrections, hence stability is a function of the maximum allowed magnitude of control correction \( \sigma_0(t) \).

In conclusion, we have demonstrated that a control scheme utilizing a point vortex model for an open flow is an effective scheme for inducing controlled capture of an extended vorticity distribution under a NS-like dynamics. The control scheme itself is simple and noise robust, and hence this result may be useful in many applications. However, to demonstrate this applicability in a realistic framework, it is still necessary to investigate the scheme using simulations with no-slip boundary conditions on the cylinder (in progress), and therefore to include the effects of the cylinder wake and secondary shedding. It is also necessary to develop the perturbation mechanism \( \sigma_0(t) \) as actual cylinder rotation, again inducing secondary shedding, to simulate a physically realizable perturbation. Finally, rather than assuming perfect knowledge of the center-of-vorticity in the control loop, we propose that an alternate method is to calculate the effective center using several fluid pressure measurements. We note that the OGY control scheme itself can also be entirely reformulated in the state space constructed purely from several pressure measurements, thereby providing a scheme which can be directly implemented in a laboratory [17].

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